Exercise 28

Use a graph of the vector field $\mathbf{F}$ and the curve C to guess whether the line integral of $\mathbf{F}$ over C is positive, negative, or zero. Then evaluate the line integral.

$$\mathbf{F}(x,y) = \frac{x}{\sqrt{x^2+y^2}}\,\mathbf{i} + \frac{y}{\sqrt{x^2+y^2}}\,\mathbf{j},$$

C is the parabola $y = 1 + x^2$ from $(-1, 2)$ to $(1, 2)$

Solution

The parameterization for the parabolic arc from $(-1, 2)$ to $(1, 2)$ is $x(t) = t$ and $y(t) = 1 + t^2$, where $-1 \le t \le 1$. As a result, the line integral becomes

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_{-1}^{1} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt \\
&= \int_{-1}^{1} \left\langle \frac{x(t)}{\sqrt{[x(t)]^2+[y(t)]^2}}, \frac{y(t)}{\sqrt{[x(t)]^2+[y(t)]^2}} \right\rangle \cdot \frac{d}{dt}\langle t, 1+t^2\rangle\,dt \\
&= \int_{-1}^{1} \left\langle \frac{t}{\sqrt{(t)^2+(1+t^2)^2}}, \frac{1+t^2}{\sqrt{(t)^2+(1+t^2)^2}} \right\rangle \cdot \langle 1, 2t\rangle\,dt \\
&= \int_{-1}^{1} \left\langle \frac{t}{\sqrt{1+3t^2+t^4}}, \frac{1+t^2}{\sqrt{1+3t^2+t^4}} \right\rangle \cdot \langle 1, 2t\rangle\,dt \\
&= \int_{-1}^{1} \left[\frac{t}{\sqrt{1+3t^2+t^4}}(1) + \frac{1+t^2}{\sqrt{1+3t^2+t^4}}(2t) \right] dt \\
&= \int_{-1}^{1} \frac{3t+2t^3}{\sqrt{1+3t^2+t^4}}\,dt \\
&= 0.
\end{aligned}$$

Because the integrand is an odd function of t and the integration interval is symmetric, the integral evaluates to zero.

A plot of the vector field $\mathbf{F}(x, y) = x/\sqrt{x^2+y^2}\,\mathbf{i} + y/\sqrt{x^2+y^2}\,\mathbf{j}$ is shown below.